Black Hole Entropy in M-theory

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1 Introduction

It has long been known [1, 2] that Einstein's equations of general relativity, when applied to black hole solutions, are analogous to the laws of thermodynamics. The crux of the analogy lies in the comparison of the following two equations. The first,

$$dM = \frac{1}{8\pi}\kappa dA \tag{1}$$

is purely a consequence of the equations of motion of general relativity and the second

$$dE = TdS. \tag{2}$$

is just the first law of thermodynamics, relating energy to entropy. In (1), M is the mass of a black hole, A the area of its horizon, and κ the surface gravity near the horizon. Einstein's equations furthermore predict that A can never decrease in any physical process, just like entropy S, thereby motivating the idea that a black hole's entropy is proportional to the area of its horizon. Hawking put this analogy on firmer ground by using semiclassical gravity techniques to show that a distant observer sees thermal radiation at temperature $T = \frac{\kappa}{2\pi}$ emanating from a black hole [3]. This observation fixes the entropy of a black hole to be $S = \frac{A}{4}$, and of course after equating a black hole's mass M with its energy E, equations (1) and (2) are exactly identical.

If we are to take seriously the above analogy between a black hole and a thermodynamic system with a given temperature and entropy, then we should be able to derive the entropy from microscopic considerations, simply by counting states. One of the great successes of string theory is that it can derive this entropy microscopically for certain classes of supersymmetric, BPS saturated black holes. The purpose of this paper is to summarize the work of [4] in which this derivation is carred out for a particular class of charged black holes in an $N = 2$ $d = 4$ supergravity theory obtained by compactifying M-theory on $S^1 \times M$ where M is a Calabi-Yau three-fold. There the macroscopic entropy $S = \frac{A}{4}$ as a function of black hole charges was calculated from supergravity considerations and compared to a microscopic entropy calculated from a dual description involving a sigma model into the moduli space of fivebranes carrying the same charges as the supergravity solution. The two entropies agree to leading order in the reciprocal of the black hole charge. Following [4], in section 1 we summarize the calculation of the macroscopic entropy, and in section 2 we summarize the microscopic calculation, exhibiting the agreement. Finally in section 3 we state carefully the regimes of validity of the calculations summarized in sections 1 and 2 and comment briefly on why the calculations agree.

2 The Macroscopic Entropy

Upon compactifying M-theory on $S^1 \times M$ one obtains an $N = 2$ $d = 4$ supergravity theory coupled to $h^{1,1}$ vector multiplets and $h^{2,1} + 1$ hypermultiplets, where $h^{1,1}$ and $h^{2,1}$ are the hodge numbers of the

Calabi-Yau three fold M. Thus we generically have a total of $h^{1,1}+1$ massless U(1) gauge fields in our low energy theory.[1] One gauge field, the graviphoton in the supergravity multiplet, comes from Kaluza-Klein reduction of the M-theory metric $g_{\mu\nu}$ on S^1, while the other $h^{1,1}$ gauge fields in the vector multiplets come from reduction of the M-theory 3-form $C_{\mu\nu\lambda}$ on the $h^{1,1}$ two-cycles of M. Let (p^Λ, q_Λ), where $\Lambda = 0, \ldots, h^{1,1}$ be the magnetic and electric charges of a black hole solution under the above $h^{1,1}+1$ gauge fields, with $\Lambda = 0$ corresponding to the graviphoton field.

Four dimensional BPS black hole solutions with the specified charges were found in [5, 6], and furthermore their macroscopic entropy was computed. In these solutions, the hypermultiplet moduli are held at fixed constants. However, the vector multiplet moduli, controlling the complexified Kahler class of M, couple to the electromagnetic fields, and hence vary as a function of radius in the black hole solution. This variation in the moduli fields is such that as one moves from asymptotic infinity to the black hole horizon, the moduli fields must satisfy a damped geodesic equation on moduli space. Because of this, regardless of the value of the moduli at asymptotic infinity, the moduli approach a fixed point in moduli space at the horizon. One can find these fixed points by searching for solutions with constant moduli, and it turns out that the fixed point depends *only* on the black hole charges (p^Λ, q_Λ). Furthermore, the area of the horizon can be expressed solely in terms of the values of the moduli at the horizon and the $N=2$ prepotential F evaluated at this same point in moduli space.

It is well known that the $N=2$ prepotential F obtained upon compactifying M-theory on $S^1 \times M$ or equivalently, type IIA on M is a function of the vector multiplet moduli given by

$$F = D_{ABC}\frac{X_A X_B X_C}{X_0}. \tag{3}$$

Here (X_0, X_A), $A = 1, \ldots, h^{1,1}$ should be thought of as projective coordinates on the vector multiplet moduli space, and D_{ABC} are intersection numbers given by

$$D_{ABC} = 6\int_M \alpha_A \wedge \alpha_B \wedge \alpha_C \tag{4}$$

where α_A form an integral basis of $H^2(M, \mathbf{Z})$. It should be emphasized that this formula is a classical approximation, and in the IIA language can receive α' corrections both perturbatively and nonperturbatively. The nonperturbative contributions come from worldsheet instantons which cause D_{ABC} to become the structure constants of the quantum cohomology ring of M. These corrections can be ignored when all vector multiplet, or Kahler moduli are large compared to string scale.

In any case, given that the black hole's charges determine the moduli at the horizon, and the area of the horizon is determined by F at the horizon, the entropy, or area, can be expressed solely in terms of the charges and the intersection numbers! This formula is given by

$$S = \sqrt{(D_{ABC}p^A p^B p^C)(q_0 + \frac{1}{12}D_C^{AB}p^C q_A q_B)} \tag{5}$$

It is this macroscopic entropy that must be explained by counting states in some microscopic description of the black hole.

3 The Microscopic Entropy

The state counting strategy used to reproduce the macroscopic entropy (5) for a given set of electromagnetic charges, is to start with M theory on $S^1 \times M \times R^{3,1}$ and construct the black hole microscopically by

[1] There will be extra massless gauge fields at special points in the vector multiplet moduli space where some 2-cycles have shrunk to zero size, but these points will not be relevant to the black hole solutions whose entropy we wish to calculate.

wrapping various $\frac{1}{2}$ BPS M-theory objects, in particular M-theory five branes, around appropriate cycles in order to carry the given set of charges. Now $\frac{1}{2}$ BPS states in space time, such as the $\frac{1}{2}$ BPS black hole solution in the $N = 2$ $d = 4$ supergravity theory, correspond to fully supersymmetric states in the low energy world volume theory of the $\frac{1}{2}$ BPS M-theory objects [7], which microscopically represent this black hole solution . So to reproduce the macroscopic entropy of a black hole with a given set of charges, one simply counts the supersymmetric states in the low energy worldvolume theory of a fivebrane wrapped so as to give rise to those charges.

So the first question to ask is what configurations of five branes gives rise to a given set of charges (p^Λ, q_Λ). In [4], the authors restrict themselves to the magnetic charge $p_0 = 0$. Nonzero p_0 would corresspond nontrivial S^1 fibrations and hence complicate the analysis. However nonzero electric charge, q_0, under the $4D$ graviphoton is simply carried by M-theory states with momentum q_0 along the S^1 circle. Furthermore magnetic charges under a gauge field obtained by wrapping the three form $C_{\mu\nu\lambda}$ on a 2-cycle Σ_A are carried by fivebranes wrapped on S^1 times a four cycle P dual to Σ_A under the intersection form. Electric charges under this same gauge field can be carried either by membranes wrapping Σ_A, or equivalently by the flux of the self dual threeform field strength $h = db$ of the two-form b which propagates on the fivebrane world volume. The flux h carries this "membrane" charge because of a Chern-Simons coupling between the bulk Ramond-Ramond three form C and the world volume flux h:

$$S_{CS} = \int_{\mathbb{R}\times S^1\times P} h \wedge C.$$

Having identified how all the charges can arise microscopically from fivebranes wrapped on a general 4-cycle P with possible h fluxes, we must now ask what the low-energy, worldvolume theory of fluctuations of this fivebrane system is. Working in the regime where the volume of M is much smaller than the radius R of the S^1, one can view the fluctuations as fluctuations in the moduli of the four-cycle P and the fluxes of the worldvolume two-form b as one travels around the circle, and through time. These fluctuations can be understood as a sigma model of maps from $S^1 \times R$ into the fivebrane moduli space. Furthermore, it is a $(0,4)$ supersymmetric conformal field theory. The $(0,4)$ SUSY is inherited from dimensionally reducing the $(0,2)$ SUSY of the fivebrane theory on $P \times S^1$, and it is conformal because there are no massive excitations. Supersymmetric states are then those states that excite only left movers, and leave the right movers in their ground state, since the supercharges are all right moving and all annhilate the right moving ground state.

Let us now count the number of supersymmetric states of a black hole with electric charge q_0 and magnetic charges p^A, where the p^A's are given by the expansion coefficients of the poincare dual of the homology class of the fivebrane world volume P in a basis w_A of $H^2(M,\mathbb{Z})$. We will deal with the more complicated case of nonzero electric charges q_A later. The entropy is given by the number of possible excitations of the sigma model with total momentum along S^1 equal to q_0. As the theory is conformal, the left moving side is characterized by a central charge c_L, and the asymptotic growth in the number of states for large $q_0 \gg c_L$ is given by

$$S_{micro} = 2\pi\sqrt{\frac{c_L q_0}{6}} \tag{6}$$

This formula can be roughly be thought of as an approximation to the logarithm of the number of ways to partition q_0 units of momentum among c_L oscillators.

Thus we have reduced the microscopic entropy to calculating the left moving central charge c_L of the theory. In this particular theory,

$$c_L = d_P + b_2^- + 3. \tag{7}$$

Namely each left moving boson contributes one to c_L with d_P bosons coming from the dimension of the moduli space of fivebrane world volumes, i.e., the moduli of P embedded in M, and b_2^- left moving scalars coming from dimensionally reducing the fivebrane two-form b on an anti-self dual two-cycle [2], and three bosons coming from transverse oscillations of the S^1 in uncompactified space.

d_P can be calculated using a standard trick in algebraic geometry, namely to consider an auxiliary holomorphic line bundle L over M with first chern class $c_1(L)$ poincare dual to P. This means that L has a section s whose zero locus is P. Now as we move around in the space of sections of L, we will trace out a family of zero loci all homologous to P. Multiplying a section by a scalar keeps fixed the zero locus. So the moduli of P can be identified with the projectivization of the space of sections of L and hence the number of real moduli is $d_P = 2(h^0(M,L)-1)$ where $h^0(M,L)$ is the complex dimension of the space of holomorphic sections of L. Furthermore $h^0(M,L)$ can be calculated using a generalization of the Riemann-Roch index theorem yielding the final answer

$$d_P = \int_M (\frac{1}{3}P^3 + \frac{1}{6}Pc_2(M)) - 2. \tag{8}$$

b_2^- can also be calculated using a combination of index theorems for the Euler character $\chi = 2 + b_2^+ + b_2^-$ and the signature $\sigma = b_2^+ - b_2^-$. The right hand side of these index theorems involve calculations of the chern classes of P. Calculations of these classes are facilitated by recognizing that L is also the normal bundle of P in M and hence yields a relation between the total chern characters:

$$c(TM|_P) = c(TP)c(L|P).$$

Using all these index theorems and relations, it is simple to arrive at the final formula

$$b_2^- = \int_M (\frac{2}{3}P^3 + \frac{5}{6}Pc_2(M)) - 1. \tag{9}$$

Now substituing (9) and (8) into (7) and further substituting (7) into (6), we obtain finally the microscopic entropy in the case of no electric charges ($q_A = 0$):

$$S_{micro} = 2\pi\sqrt{\frac{(P^3 + c_2P)q_0}{6}} \tag{10}$$

For large black hole charges, $P^3 \gg c_2P$, this can be expanded as

$$S_{micro} = 2\pi\sqrt{\frac{P^3q_0}{6}} + c_2P\frac{\pi}{6}\sqrt{\frac{q_0}{P^3}} + \ldots \tag{11}$$

Recognizing that when P is thought of as a two-form and expanded in a basis, $P = p^A\alpha_A$, then using definition (4), $P^3 = 6D_{ABC}p^Ap^Bp^C$, and we see that the first term in the expansion (11) matches exactly the macroscopic entropy (5) when electric charges q_A are zero! The remaining terms may be viewed as corrections to the macroscopic entropy, whose calculation was based on the purely classical prepotential F in (3). Of course this classical approximation cannot be expected to be exact. Indeed the second term in the expansion captures a one-loop correction to the macroscopic entropy due to an R^4 term in the original $d = 11$ M-theory action [4].

The inclusion of electric charges, although not hard to understand, is slightly more complicated to explain, and in an effort to keep this paper shorter than the paper it aims to summarize, we will give a

[2] The self-duality condition on the field strength $h = db$ forces scalars obtained from b wrapped on a self-dual 2-cycle to be right moving and scalars obtained from b wrapped on a anti-self dual 2-cycle to be left moving.

very brief treatment of it. Recall that membrane charge is carried by the flux of h through 3-cycles of the form $S^1 \times \Sigma_A$, with Σ_A a 2-cycle in the fivebrane world volume, i.e. in P. Further recall that a scalar in the sigma model is obtained by wrapping the two form b on the cycle Σ_A by making the ansatz $b = X^A(\sigma, \tau)\alpha_A$ (no summation). Here α_A is a two-form dual under integration to the two-cycle Σ_A. Forcing $h = db$ (a formula which only applies locally) to have nontrivial three flux through $S^1 \times \Sigma_A$ then forces the scalar X^A to have nontrivial winding around S^1. This is the result we are after. Electric charges q^A are carried in the sigma model, by states with nontrivial winding in the scalars X^A. A nonzero winding number of the scalars, shifts the ground state energy and momentum of the effective two-dimensional theory. This in turn leads to an effective shift in q_0 to a new $\tilde{q}_0$ given by

$$\tilde{q}_o = q_0 + \frac{1}{12} D^{AB}_C p^C q_A q_B. \tag{12}$$

Substituting $\tilde{q}_0$ in place of q_0 in (11) yields to lowest order the classical macroscopic entropy (5) with nonzero electric charges q_A.

4 Regimes of Validity

The above agreement works well, but unlike in [4], we have up till now been relatively glib about the regimes in moduli space where each entropy calculation, whether macroscopic or microscopic, is valid. It is time to be more precise about these regimes of validity in order to justify the approximations made in sections 1 and 2 and to understand the origin of corrections to either calculation. The parameters we have to play with are the black hole electric charge $\tilde{q}_0^3$, magnetic charge P^3, the radius R of the S^1, and the volume V_M of the Calabi-Yau M.

The supergravity approximation used to calculate (5) works well when all scales are large in 11 dimensional planck units, including V_M and R. V_M is a hypermultiplet scalar which is a freely adjustable constant throughout the solution. However $R^3 V_M$ is a vector multiplet whose value, via the damped geodesic equations, is forced to attain $\sqrt{\frac{\tilde{q}_0^3}{P^3}}$ at the black hole horizon. Thus we require $\tilde{q}_0^3 \gg P^3$, which incidentially justifies the approximation made in (6). Also demanding that the black hole supergravity solution is weakly curved means requiring large magnetic charges, which via BPS condition translates into large mass or weak curvature. In essence we require $P^3 \gg V_M$.

In contrast, the validity of the dual perspective, or the low energy sigma model theory of fivebrane fluctuations, depends on the fivebranes being far apart from each other and on their being embedded in a large space so one can keep the low energy description of the five brane. This corresponds to small charges, and large Calabi-Yau volumes, or $P^3 \ll V_M$. This is the opposite of the requirement of the validity of the supergravity calculation, but as mentioned before, V_M is in a hypermultiplet and can be varied continuously from small to large without affecting the number of BPS states, so the conflicting requirements do not pose a problem. However, for the fluctuations of the fivebrane to reduce to a sigma model, it was required that $R^6 \gg V_M$. Satisfying this means moving around in the vector multiplet moduli space away from the region of validity of the supergravity approximation, and so an implicit assumption in this work is that the number of BPS states do not change as one does this. This assumption is justified in hindsight by the relatively good agreement between the two calculations.

Thus the moral of this work, is that the SAME 1/2 BPS object with a given set of charges (p^Λ, q_Λ) is best described as a black hole supergravity solution in one region of moduli space, and by a sigma model of the low energy fluctuations of a collection of fivebranes in another region of moduli space. One can calculate entropies in both descriptions and hope for a match. The fact that the match works out so well hinges on the BPS properties of the states being counted, and their relative stability under changes of moduli. However the microscopic calculation hints at a host of g_s and α' corrections (in IIA language)

to the macroscopic entropy, and, as they say in [4], it would be interesting to reproduce these corrections to the entropy, or area of the black hole horizon as a supergravity solution, from corrections to the supergravity equations themselves.

References

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